MATHEMATICS

THE RADON-NIKODYM THEOREM. I

BY

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1. *Introduction*

The present paper [1]) is devoted to several remarks on the Radon-Nikodym theorem, in the σ-finite as well as in the non-σ-finite case, and in some of the sections the emphasis will be more on methods than on new results. The Radon-Nikodym theorem is often cited only for the σ-finite case, and there is even a remark in § 14 of S. SAKS' well-known treatise [3], with an example as illustration, which may easily be misinterpreted to mean that σ-finiteness of the underlying measure is a necessary condition for the theorem to hold. This is, however, not true. It has been shown, in fact, by I. E. SEGAL [4] that another property of the measure, called localizability, is necessary and sufficient in order that the Radon-Nikodym theorem is valid, and it is easy to give examples of localizable measures which fail to be σ-finite. It is, therefore, rather surprising at first to see that in Saks' "counterexample" the measure is localizable in Segal's sense (it is discrete measure, i.e. the measure of each set E is the number of points in E), and yet the theorem fails to hold. Evidently, there must then be some other phenomenon which is responsible for this failure. We shall return to Saks' example in sec. 4, and try to show why it does not work. It should be added that in my own textbook on integration [6] the same example is presented (in § 31) with a similar confusing remark.

In his paper referred to above, I. E. Segal is interested in a number of other properties besides localizability and the property that the Radon-Nikodym theorem holds, and he proves the equivalence of six of these properties by "moving along the circumference of a circle". Moreover, in these proofs use is made of an imbedding theorem asserting that every measure space, either localizable or not, may be imbedded (measure preserving and isomorphic in a certain algebraic sense) into another measure space which is always localizable. It is not surprising, therefore, that the route from localizability to the Radon-Nikodym theorem and backwards is not easy to retrace. One of the purposes of

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this paper is to show that a direct route (without imbedding) exists such that, on the way, it remains intuitively clear what one is doing and what one should do next. As may be expected in the case of a non-σ-finite measure, however, one should have no objection against using Zorn's lemma. Also the reader is warned not to expect that proofs which formerly needed a number of pages can now all at once be condensed into a few lines.

2. *Preliminaries on integration*

We shall assume the reader to be familiar with the notations and contents of the first seven chapters of [6]. By a measure μ on the semi-ring Γ (in the applications often a ring or a σ-ring) of subsets E of the non-empty point set X we mean therefore a non-negative and countably additive set function $\mu(E)$, defined for all $E \in \Gamma$ and such that $\mu(\phi)=0$, where ϕ is the empty set (μ may assume the value $+\infty$). By applying the Carathéodory procedure this measure may be extended so as to become defined on the σ-ring of all μ-measurable sets, and if we speak in the following simply about a measure μ in X, then it is assumed that μ is already the thus extended measure. Furthermore, if L is a linear vector lattice of real bounded functions $f(x)$ on X (i.e. L is a linear collection such that $f, g \in L$ implies $\max(f, g) \in L$ and $\min(f, g) \in L$), then any (finitevalued) non-negative linear functional $\mathscr{I}(f)$ on L, having the additional property that $f_n(x) \downarrow 0$ on X implies $\mathscr{I}(f_n) \downarrow 0$, is called an elementary integral on L. This elementary integral $\mathscr{I}(f)$ may be extended, so as to become defined finally on a class of functions including L as a subclass, by a procedure similar to the Carathéodory procedure for measures. This is done first for non-negative functions, and any function $f(x) \geqslant 0$ for which the extended $\mathscr{I}(f)$ is thus defined (either as a finite number or as being equal to $+\infty$) is called a non-negative $\mathscr{I}$-measurable function. The extension of $\mathscr{I}(f)$ to non-positive functions is then immediate, and the real function $f=f^{+}+f^{-}$, where $f^{+}=\max(f, 0)$ and $f^{-}=\min(f, 0)$, is called $\mathscr{I}$-measurable whenever f^{+} and f^{-} are $\mathscr{I}$-measurable in the already defined sense. The integral of this f is defined by $\mathscr{I}(f)=\mathscr{I}(f^{+})+\mathscr{I}(f^{-})$, unless $\mathscr{I}(f^{+})=+\infty$ and $\mathscr{I}(f^{-})=-\infty$ hold simultaneously. In this exceptional case $\mathscr{I}(f)$ is left undefined. The class of all functions f for which the extended $\mathscr{I}(f)$ is thus defined is the class of all $\mathscr{I}$-integrable functions, and the subclass of all f satisfying $-\infty<\mathscr{I}(f)<+\infty$ is the class of $\mathscr{I}$-summable functions. As stated before, the initial domain of definition of $\mathscr{I}(f)$ is denoted by L; the classes of measurable and summable functions will be denoted by M and L_1 respectively. Evidently we have $L \subset L_1 \subset M$. The subclasses of all non-negative functions in L, L_1 and M will be denoted by L^{+}, $L_1{}^{+}$ and M^{+} respectively.

The thus extended integral $\mathscr{I}(f)$ is called a Daniell integral or a Daniell–Stone integral. The extension procedures of P. J. Daniell and M. H. Stone cannot immediately be described as copies of Carathéodory's procedure, but the extensions obtained by means of these somewhat different procedures are all the same.

We recall some theorems.

(a) It is an important fact in measure theory that a set E is μ-measurable if and only if the intersection $E \cap A$ is μ-measurable for every set $A \in \Gamma$ of finite μ-measure, where Γ is the semi-ring (ring, σ-ring) which served as the initial domain of definition of μ. The parallel for integrals is that the function $f(x) \geqslant 0$ is $\mathscr{I}$-measurable if and only if $\min(f, l)$ is $\mathscr{I}$-measurable (and hence $\mathscr{I}$-summable) for every function $l \in L^+$.

(b) Let μ be a measure on the ring Γ of subsets of X (we do not assume, therefore, that μ is already extended as much as possible). Then the collection of all real step functions $s(x) = \sum_1^p c_n \chi_{A_n}(x)$ (where p is finite but variable, c_n real and finite, and χ_{A_n} the characteristic function of the set $A_n \in \Gamma$ of finite μ-measure) is a linear vector lattice L_s, and $\mathscr{I}(s) = \sum_1^p c_n \mu(A_n)$ is an elementary integral on L_s. The corresponding extended integral $\mathscr{I}(f)$ is now called the Stieltjes–Lebesgue integral with respect to μ, and denoted by $\mathscr{I}(f) = \int f \, d\mu$. Given the general Daniell integral $\mathscr{I}(f)$, extension of an elementary integral initially defined on the linear vector lattice L, this $\mathscr{I}(f)$ automatically induces a measure ν in X, in the following way: The collection of all sets E with $\mathscr{I}$-measurable characteristic function χ_E is a σ-ring Γ, and $\nu(E) = \mathscr{I}(\chi_E)$ is a measure on Γ. The question may be raised under which conditions the Stieltjes–Lebesgue integral $\int f \, d\nu$ is again the given $\mathscr{I}(f)$. It turns out that $\mathscr{I}(f) = \int f \, d\nu$ holds if and only if χ_X, the characteristic function of the whole set X, is $\mathscr{I}$-measurable (in other words, if and only if $\min(f, 1) \in M^+$ for all $f \in L^+$). For the proof we refer to [6], § 17, Theorem 7 and Exercise 2.

(c) Let $\mathscr{I}(f)$ be such a Stieltjes–Lebesgue integral, and let ν be the measure induced by $\mathscr{I}(f)$ on the σ-ring Γ of all $\mathscr{I}$-measurable sets. It is true then that Γ is already the σ-ring of all ν-measurable sets. In other words, if the Carathéodory extension procedure for measures is applied to the measure ν on Γ, then no proper extension is obtained (cf. [6], § 17, Theorem 8).

(d) Finally, let μ be a measure initially defined on the ring Γ, and let $\mathscr{I}(f)$ be the integral obtained by extension of the elementary step function integral with respect to μ on Γ. The integral $\mathscr{I}(f)$ induces the measure ν on the σ-ring Λ_ν of all $\mathscr{I}$-measurable sets. On the other hand, by applying the extension procedure for measures to μ on Γ, we obtain the extended measure μ on the σ-ring Λ_μ of all μ-measurable sets. Then $\Lambda_\mu = \Lambda_\nu$, and $\mu = \nu$ on $\Lambda_\mu = \Lambda_\nu$ (cf. [6], § 17, Theorem 9). It is in view of this theorem that the notation $\int f \, d\mu$ (or $\int f \, d\nu$) has an unambiguously defined meaning.

(e) In the following, the characteristic function of any set $E \subset X$ will always be denoted by χ_E. Furthermore, any function f such that

$\mathscr{I}(|f|)=0$ will be called an $\mathscr{I}$-null function, and any set $E \subset X$ such that χ_E is an $\mathscr{I}$-null function will be called an $\mathscr{I}$-null set. If f_1 and f_2 are functions such that f_1-f_2 is an $\mathscr{I}$-null function, then f_1 and f_2 are called $\mathscr{I}$-almost equal. The relation to be almost equal is an equivalence relation, and f_1 and f_2 are almost equal if and only if the set on which $f_1(x) \neq f_2(x)$ is a null set. Changing the values of an $\mathscr{I}$-integrable function f on an $\mathscr{I}$-null set does not affect the value of $\mathscr{I}(f)$, and for this reason almost equal functions are often identified, i.e. speaking about a function f one means the entire equivalence class of functions almost equal to f. Usually this does no harm, but there are cases when it is desirable to distinguish carefully between a function f and the equivalence class to which f belongs.

(f) Finally, given the Stieltjes–Lebesgue integral $\mathscr{I}(f)=\int f\, d\mu$, the set $E \subset X$ is said to be of σ-finite μ-measure whenever E is an at most countable union of sets of finite μ-measure. If $f(x)$ is $\mathscr{I}$-summable, then the set on which $f(x) \neq 0$ is easily proved to be of σ-finite μ-measure. In particular, if $l(x) \in L$ (L is the initial domain of definition of the integral; the functions of L are not necessarily the μ-step functions), then the set $\{x : l(x) \neq 0\}$ is of σ-finite μ-measure. If the entire set X is of finite or σ-finite μ-measure, then μ is called a finite or σ-finite measure respectively. Ordinary Lebesgue measure on the real line is σ-finite, but discrete measure on the real line (each point is of measure one) is non-σ-finite.

If no confusion is likely to occur we shall sometimes write $\mathscr{I}f$ instead of $\mathscr{I}(f)$ in the next sections.

3. *Absolute continuity*

In the Radon-Nikodym theorem two integrals, taken over the same point set X, are compared. Let, therefore, $\mathscr{I}f$ and $\mathscr{J}f$ be Daniell integrals over X, extensions of elementary integrals having the same linear vector lattice L as initial domain of definition. Note that, after the extension, the classes of $\mathscr{I}$-measurable and $\mathscr{J}$-measurable functions are not necessarily the same. The integral $\mathscr{J}f$ is now called $\mathscr{I}$-absolutely continuous whenever every $\mathscr{I}$-null set is also a $\mathscr{J}$-null set, in other words, whenever for every set $E \subset X$, satisfying $\mathscr{I}\chi_E=0$, the number $\mathscr{J}\chi_E$ exists and satisfies $\mathscr{J}\chi_E=0$. It follows then immediately that every $\mathscr{I}$-null function is also a $\mathscr{J}$-null function, and every $\mathscr{I}$-measurable function is also $\mathscr{J}$-measurable (cf. [6], § 31, in particular part (a) of the proof of Theorem 1).

Assume now that $\mathscr{I}f$ is a Stieltjes–Lebesgue integral and $\mathscr{J}f$ is $\mathscr{I}$-absolutely continuous. Then $\mathscr{J}f$ is also a Stieltjes–Lebesgue integral. Indeed, since $\mathscr{I}f$ is a Stieltjes–Lebesgue integral, the characteristic function χ_X is $\mathscr{I}$-measurable (cf. sec. 2(b)), and hence $\mathscr{J}$-measurable by what has been observed in the preceding paragraph. But this (cf. sec. 2(b) again) is sufficient to ensure that $\mathscr{J}f$ is a Stieltjes–Lebesgue integral.

Lemma 3.1. *Let $\mathscr{I}f = \int f\,d\mu$ and $\mathscr{J}f = \int f\,d\nu$ be Stieltjes–Lebesgue integrals over X, extensions of elementary integrals initially defined on the same linear vector lattice L (we do not assume that one of the integrals is absolutely continuous with respect to the other). Then any ν-measurable set of σ-finite ν-measure is included in some μ-measurable set of σ-finite μ-measure (and the same with μ and ν interchanged). In particular, the measures μ and ν are simultaneously σ-finite or non-σ-finite.*

Proof. It is evidently sufficient to prove that any set $E \subset X$ of finite ν-measure is included in a μ-measurable set of σ-finite μ-measure. Let, therefore, $\nu(E) < \infty$, i.e. $\mathscr{J}\chi_E < \infty$. Then there exists a function $s(x) \geqslant \chi_E(x)$ such that $s(x)$ is the limit of a pointwise non-decreasing sequence of functions $l_n(x) \in L^+$, and $\mathscr{J}s < \infty$ (the function $s(x)$ is a σ-function in the terminology of [6]). Since all sets $F_n = \{x : l_n(x) > 0\}$ are of σ-finite μ-measure (in view of the $\mathscr{I}$-summability of all l_n; cf. sec. 2(f)), the same holds for $F = \{x : s(x) > 0\} = \bigcup_1^\infty F_n$. Observing that $E \subset F$ in view of $s \geqslant \chi_E$, we obtain the desired result.

4. *The Radon-Nikodym theorem in the σ-finite case*

The Radon-Nikodym theorem for the case of σ-finite measures may now be formulated as follows:

Theorem 4.1. (*Radon-Nikodym theorem*; *integral version*). *Let $\mathscr{I}f$ and $\mathscr{J}f$ be Stieltjes–Lebesgue integrals over X, extensions of elementary integrals on the same initial domain of definition L. Let $\mathscr{J}f$ be $\mathscr{I}$-absolutely continuous, and $\mathscr{J}\chi_X < \infty$ (the last assertion says, therefore, that the measure induced in X by the integral $\mathscr{J}$ is a finite measure). Then the measure induced in X by the integral $\mathscr{I}$ is σ-finite, and there exists an $\mathscr{I}$-summable function $f_0(x) \geqslant 0$ on X such that, for any function f, this f is $\mathscr{J}$-summable if and only if ff_0 is $\mathscr{I}$-summable, and $\mathscr{J}f = \mathscr{I}(ff_0)$ for any such f. The function $f_0 \geqslant 0$, satisfying these conditions, is $\mathscr{I}$-uniquely determined (that is, if $f_0' \geqslant 0$ satisfies the same conditions, then $f_0 - f_0'$ is an $\mathscr{I}$-null function).*

Proof. It follows from Lemma 3.1 that the measure induced in X by the integral $\mathscr{I}$ is σ-finite. The proof of the other assertions may be found e.g. in [6], § 31, Theorem 1.

The Radon-Nikodym theorem is often stated not in terms of the Stieltjes–Lebesgue integrals $\mathscr{I}f$ and $\mathscr{J}f$, but in terms of the measures μ and ν induced in X by $\mathscr{I}f$ and $\mathscr{J}f$ respectively. The formulation is then made such that the integrals $\mathscr{I}$ and $\mathscr{J}$ themselves have disappeared altogether from the hypotheses. Furthermore, in the older formulations of this kind the attention was entirely restricted to the $\mathscr{I}$-measurable sets. These sets are automatically $\mathscr{J}$-measurable whenever $\mathscr{J}$ is $\mathscr{I}$-absolutely continuous. There may be $\mathscr{J}$-measurable sets, however, which fail to be $\mathscr{I}$-measurable, and they were not mentioned in the older work. In order to obtain a more general formulation which takes care of these sets too, assume that μ and ν are measures in X, extensions of measures initially defined on the same ring Λ of subsets of X. The measure ν is then called μ-absolutely continuous whenever $\mu(E)=0$ implies $\nu(E)=0$, i.e. whenever, after extension of the measures by means of the Carathéodory procedure, any μ-measurable set of μ-measure zero is ν-measurable and of ν-measure zero. Furthermore, given the measure μ in X, any function $f(x)$ on X will be called μ-measurable or μ-summable whenever f is measurable or summable respectively with respect to the integral $\int f\,d\mu$.

Theorem 4.2. (*Radon-Nikodym theorem*; *measure version*). *Let μ and ν be measures in X, extensions of measures initially defined on the same ring Λ of subsets of X, such that $\mu(E)$ and $\nu(E)$ are finite for all $E \in \Lambda$. Let ν be μ-absolutely continuous, and $\nu(X)$ finite. Then the measure μ is σ-finite, and there exists a μ-summable function $f_0(x) \geqslant 0$ such that, for any function f, this f is ν-summable if and only if ff_0 is μ-summable, and $\int f\,d\nu=\int ff_0\,d\mu$ for any such f. In particular, $\nu(E)=\int f_0\chi_E d\mu$ for all ν-measurable sets $E \subset X$ (and hence, in the usual notation, $\nu(E)=\int_E f_0 d\mu$ for all μ-measurable sets $E \subset X$). Finally, the function $f_0 \geqslant 0$, satisfying these conditions, is μ-uniquely determined (that is, if $f_0' \geqslant 0$ satisfies the same conditions, then f_0-f_0' is a μ-null function, i.e. the set $\{x : f_0-f_0' \neq 0\}$ is a μ-null set).*

We shall prove that the two versions are equivalent. Assume first that the integral version holds, and let the hypotheses of the measure version be satisfied. In this case, denote by L the collection of all step functions $f(x)=\sum_1^p c_n\chi_{E_n}(x)$ with c_n real (and finite) and $E_n \in \Lambda$ for $n=1, \ldots, p$, and let $\mathscr{I}f=\sum_1^p c_n\mu(E_n)$ and $\mathscr{J}f=\sum_1^p c_n\nu(E_n)$ on L. Note that, in view of the hypotheses, $\mathscr{I}f$ and $\mathscr{J}f$ are finite on L. It follows that $\mathscr{I}f$ and $\mathscr{J}f$ are elementary integrals on the linear vector lattice L. By sec. 2(d) the measures induced in X by the extended integrals $\mathscr{I}f$ and $\mathscr{J}f$ are the same as the measures obtained by immediately applying the extension procedure for measures to μ and ν on Λ, so the μ-absolute continuity of ν implies the $\mathscr{I}$-absolute continuity of $\mathscr{J}$. Hence, all hypotheses for applying the integral version of the theorem to $\mathscr{I}f$ and $\mathscr{J}f$ are satisfied. It follows that the measure induced in X by $\mathscr{I}f$ is σ-finite, i.e. μ is σ-finite. Furthermore, there exists an $\mathscr{I}$-unique and $\mathscr{I}$-summable (that is, μ-unique and μ-summable) function $f_0 \geqslant 0$ such that any f is $\mathscr{J}$-summable (that is, ν-summable) if and only if ff_0 is $\mathscr{I}$-summable (that is, μ-summable), and $\mathscr{J}f=\mathscr{I}(ff_0)$ for any such f, i.e. $\int f\,d\nu=\int ff_0 d\mu$ for any such f.

Assume now, conversely, that the measure version holds, and let the integrals $\mathscr{I}f$ and $\mathscr{J}f$, satisfying the hypotheses of the integral version, be given. In this case, denote by μ and ν the measures induced by $\mathscr{I}f$ and $\mathscr{J}f$ respectively, and let Λ be the ring of all sets of finite μ-measure. It follows from sec. 2(b) that by applying the extension procedure for integrals to the elementary step function integral with respect to μ on Λ the given integral $\mathscr{I}f$ is reobtained. Similarly, by extending the elementary step function integral with respect to ν on the ring Λ_1 of all sets of finite ν-measure, we obtain $\mathscr{J}f$. By the $\mathscr{I}$-absolute continuity of $\mathscr{J}$ any set $E \in \Lambda$ is ν-measurable, so $\Lambda \subset \Lambda_1$ (the finiteness of $\nu(E)$ for any $E \in \Lambda$ follows from $\mathscr{J}\chi_X < \infty$). It may be, however, that Λ_1 is properly larger than Λ. It will simplifly the situation if we prove first that $\mathscr{J}f$ is already obtained by extending the elementary step function integral with respect to ν on Λ. Evidently, this elementary integral is the restriction of $\mathscr{J}f$ to the collection of μ-step functions. Since any non-negative $f \in L$ (we recall that L is the initial domain of definition of $\mathscr{I}$ and $\mathscr{J}$) is $\mathscr{I}$-summable, there exists a sequence f_n of μ-step functions such that $0 \leqslant f_n \uparrow f$; hence $\mathscr{J}f_n \uparrow \mathscr{J}f$, and this shows that $\mathscr{J}f$ on L, and hence the entire extended integral $\mathscr{J}f$, is reobtained by starting from the μ-step functions. It follows then by sec. 2(d) that the measure ν may be obtained by applying the extension procedure for measures to its own restriction to the sets of Λ. In other words, the measures μ and ν in X may both be regarded as extensions of their own restrictions to the sets of Λ. All hypotheses of the measure version are satisfied therefore for μ and ν, and the desired integral version for $\mathscr{I}$ and $\mathscr{J}$ follows.

The first assumption in the measure version is that the measures μ and ν may be regarded as extensions of measures initially defined on the same ring Λ of subsets of X, such that $\mu(E)$ and $\nu(E)$ are finite for every $E \in \Lambda$. The question may be raised to which extent the finiteness assumption is essential for the validity of the theorem. The assumption that $\nu(E)$ is finite for all $E \in \Lambda$ is subsumed under the further and stronger assumption that $\nu(X)$ is finite; the question remains, therefore, whether the theorem is still true if it is no longer assumed that $\mu(E) < \infty$ for all $E \in \Lambda$. Assume, therefore, merely that μ and ν are measures in X, extensions of measures initially defined on the same ring Λ of subsets of X. Let ν be μ-absolutely continuous, and $\nu(X) < \infty$. Similarly as before, we let L be the collection of all step functions $f(x) = \sum_1^p c_n \chi_{E_n}(x)$ with $E_n \in \Lambda$, but now with the additional condition that $\mu(E_n) < \infty$ for $n = 1, \ldots, p$, and we set $\mathscr{I}f = \sum_1^p c_n \mu(E_n)$ on L. Then it remains true, by sec. 2(d), that the measure induced in X by the extended integral $\mathscr{I}f$ is the same as the given measure μ. It is not certain, however, that if the elementary integral $\mathscr{J}f$ is now defined on L by $\mathscr{J}f = \sum_1^p c_n \nu(E_n)$, then the measure induced in X by the extended $\mathscr{J}f$ is the given measure ν. The theorem cited in sec. 2(d) ensures only that if $\hat{\mathscr{J}}f = \sum_1^p c_n \nu(E_n)$ on the collection $\hat{L}$ of all functions $f(x) = \sum_1^p c_n \chi_{E_n}(x)$ with $E_n \in \Lambda$ (regardless of whether $\mu(E_n)$ is finite or infinite), then the measure induced in X by the extended $\hat{\mathscr{J}}f$ is the given measure ν. Due to the additional condition that $\mu(E) < \infty$ for the sets E occurring in the step functions of L, the collection L may be considerably smaller than $\hat{L}$.

This is exactly what happens in Saks' "counterexample" referred to in sec. 1. In this example we have $X = [0, 1]$, and Λ is the collection of all

Lebesgue measurable subsets of X. The measure μ is the discrete measure on Λ (i.e., the measure $\mu(E)$ of $E \in \Lambda$ is the number of points in E), and ν is Lebesgue measure on Λ. Extension yields for the extended μ the discrete measure in X (each subset of X is μ-measurable), and the given ν is already extended as much as possible. Obviously, ν is μ-absolutely continuous, and $\nu(X)=1$. By means of an elementary argument it is easily seen that the Radon-Nikodym theorem fails to hold. Note that the collection L referred to in the preceding paragraph consists here of all (finite valued) functions differing from zero only at a finite number of points, whereas $\hat{L}$ consists of all Lebesgue measurable step functions. The elementary Lebesgue integral $\hat{\mathscr{J}}f$ on $\hat{L}$ yields upon extension the ordinary Lebesgue integral, whereas the elementary Lebesgue integral $\mathscr{J}f$ on L is identically zero, and so remains zero for all $\mathscr{J}$-summable functions when extended. The failure of the Radon-Nikodym theorem in this example is due, therefore, to the fact that L is too small compared to $\hat{L}$. Expressed differently, the theorem fails because the Lebesgue integral and the integral with respect to discrete measure cannot be regarded as extensions of elementary integrals having a common domain of definition.

5. *Abstract Borel sets*

We assume, once more, that the Stieltjes–Lebesgue integrals $\mathscr{I}f=\int f\,d\mu$ and $\mathscr{J}f=\int f\,d\nu$ are extensions of elementary integrals with the same initial domain of definition. As observed in sec. 2(b), $\mathscr{I}f$ may also be regarded as the extension of the elementary μ-step function integral and, similarly, $\mathscr{J}f$ is the extension of the elementary ν-step function integral. Unfortunately, without any further assumptions neither of these two classes of step functions need include the other, and the question may be raised whether there exists a suitable common subclass of step functions such that $\mathscr{I}f$ and $\mathscr{J}f$ are the extensions of their own restrictions to this subclass. We shall prove that, at least for the case that $\mu(X)$ and $\nu(X)$ are finite, there exists a subclass of the desired kind, and this will turn out to be useful in the theory of infinite product integrals.

We recall that a σ-ring of subsets of X, containing the set X itself as one of its members, is usually called a σ-field. Given now the Stieltjes–Lebesgue integral $\mathscr{I}f=\int f\,d\mu$ over X, having the linear vector lattice L as initial domain of definition, we let Λ be the smallest σ-field of subsets of X such that Λ contains all sets of the form $\{x : f(x) > a\}$, where f varies through L^+ and a varies through the positive numbers. We shall say that the sets of Λ are the *Borel sets* with respect to L. Note that Λ depends only on L, and not on the integral $\mathscr{I}f$. Any finite linear combination of $\mathscr{I}$-summable characteristic functions of Borel sets will be called a Borel $\mathscr{I}$-step function.

Theorem 5.1. (a) *Any Borel set is μ-measurable.*

(b) *The restriction of $\mathscr{I}f$ to the Borel $\mathscr{I}$-step functions is an elementary* *integral, and by applying the extension procedure for integrals to this elementary integral the given integral $\mathscr{I}f$ is reobtained.*

(c) *Any μ-summable subset of X is included in and μ-almost equal to a Borel set.*

Proof. (a) For any $a > 0$ and any $f \in L^+$, it is well-known that $E = \{x : f(x) > a\}$ is μ-measurable (cf. [6], § 17, Theorem 2). Hence, the σ-field of all μ-measurable sets contains all such sets E. It follows that the smallest σ-field Λ containing all such E is included in the σ-field of the μ-measurable sets. In other words, any Borel set is μ-measurable.

(b) Evidently, the restriction of $\mathscr{I}f$ to the Borel $\mathscr{I}$-step functions is an elementary integral. Furthermore, given $f \in L^+$, there exists a sequence f_n of Borel $\mathscr{I}$-step functions such that $f_n \uparrow f$ on X(cf. [6], § 17, Theorem 4), so $\mathscr{I}f_n \uparrow \mathscr{I}f$. This shows that, starting from the elementary integral $\mathscr{I}f$ on the Borel $\mathscr{I}$-step functions and extending it, the integral $\mathscr{I}f$ on L^+ is reobtained on the way; hence, the given integral $\mathscr{I}f$ is reobtained.

(c) It follows from the result in the preceding paragraph, by means of the theorem cited in sec. 2(d), that by applying the Carathéodory extension procedure to the measure μ on the σ-field Λ of Borel sets the given measure μ is reobtained. Hence, given the μ-summable set E (that is, $\mu(E) < \infty$), there exists a descending sequence of sets $O_n \supset E$, each O_n a countable union of sets of Λ, such that $O = \bigcap_1^\infty O_n$ satisfies $\mu(O - E) = 0$. But we have $O \in \Lambda$, since Λ is a σ-field; hence E is included in and μ-almost equal to the Borel set O.

Theorem 5.2. *Consider the class of all Stieltjes–Lebesgue integrals having the linear vector lattice L as initial domain of definition, and inducing finite measures in X. Let Λ be the σ-field of the Borel sets (with respect to L), and L_B the corresponding linear vector lattice of all Borel step functions (i.e., L_B consists of all finite linear combinations of characteristic functions of sets of Λ). Then each integral in the class referred to above is the extension of its own restriction to L_B.*

Proof. Follows immediately from the preceding theorem by observing that if $\mathscr{I}$ is an arbitrary integral in the class referred to, then any $f \in L_B$ is $\mathscr{I}$-summable in view of $\mathscr{I}\chi_X < \infty$.

We shall also need the following theorem:

Theorem 5.3. *Let $\mathscr{I}f = \int f \, d\mu$ be the extension of an elementary integral initially defined on the linear vector lattice L, such that the induced measure μ is σ-finite, and let the function $f_0(x) \geqslant 0$ be $\mathscr{I}$-summable. Then:*

(a) *$\mathscr{J}f = \mathscr{I}(ff_0)$ is an elementary integral on L;*

(b) *If this elementary integral is extended, the extended $\mathscr{J}f$ is $\mathscr{I}$-absolutely continuous, and $\mathscr{J}\chi_X < \infty$;*

(c) *By the Radon–Nikodym theorem there exists now an $\mathscr{I}$-summable function $f_0^*(x) \geqslant 0$ such that $\mathscr{J}f = \mathscr{I}(ff_0^*)$ holds for all $\mathscr{J}$-summable f.*

This $f_0{}^*$ and the given f_0 are $\mathscr{I}$-almost equal, i.e. the equality $\mathscr{J}f = \mathscr{I}(ff_0)$ continues to hold after extension of $\mathscr{J}$.

Proof. (a) This is evident (note that $\mathscr{I}(ff_0)$ exists for any $f \in L$ since f is a bounded function).

(b) Evidently, $\mathscr{J}s = \mathscr{I}(s\,f_0)$ holds for any σ-function $s(x)$, where by a σ-function $s(x)$ we mean the limit of a pointwise non-decreasing sequence of functions of L^+. But then $\mathscr{J}g = \mathscr{I}(g\,f_0)$ holds for any $\mathscr{J}$-summable σ_δ-function g, where by a σ_δ-function we mean the limit of a pointwise non-increasing sequence of σ-functions. Since for any non-negative $\mathscr{J}$-null function f there exists a σ_δ-function $g \geqslant f$ such that $\mathscr{J}g = 0$, it follows immediately that $\mathscr{I}(ff_0) \leqslant \mathscr{I}(gf_0) = \mathscr{J}g = 0$, so $\mathscr{J}f = \mathscr{I}(ff_0) = 0$. Hence, since any non-negative $\mathscr{J}$-summable function is the difference of a σ_δ-function and a non-negative $\mathscr{J}$-null function, the equality $\mathscr{J}f = \mathscr{I}(ff_0)$ holds for any $\mathscr{J}$-summable function f.

Next, we prove that $\mathscr{J}f = \mathscr{I}(ff_0)$ holds for any σ_δ-function f, whether $\mathscr{J}$-summable or not. Let k be a σ-function such that $k \geqslant f$, and let $k_n \in L^+$, $k_n \uparrow k$. Then the functions $p_n = \min(k_n, f)$ are $\mathscr{J}$-summable (since k_n is $\mathscr{J}$-summable and f is $\mathscr{J}$-measurable), so $\mathscr{J}p_n = \mathscr{I}(p_n f_0)$. Hence $\mathscr{J}f = \mathscr{I}(ff_0)$ in view of $p_n \uparrow f$.

Assume now that $f \geqslant 0$ is an $\mathscr{I}$-null function. Then there exists a σ_δ-function $h \geqslant f$ such that $\mathscr{I}h = 0$, so $\mathscr{I}(hf_0) = 0$ holds as well. But $\mathscr{J}h = \mathscr{I}(hf_0)$ since h is a σ_δ-function, so $\mathscr{J}h = 0$. On account of $0 \leqslant f \leqslant h$ it follows that $\mathscr{J}f = 0$, and this shows that $\mathscr{J}$ is $\mathscr{I}$-absolutely continuous.

Observing now that $\mathscr{I}$-measurability implies $\mathscr{J}$-measurability by the $\mathscr{I}$-absolute continuity of $\mathscr{J}$, it follows in the same way as for a σ_δ-function that $\mathscr{J}f = \mathscr{I}(ff_0)$ holds for any $\mathscr{I}$-summable $f \geqslant 0$ (both sides in the equality may be $+\infty$). Since by hypothesis $\chi_X = \sum_1^\infty \chi_{A_n}$, where each χ_{A_n} is $\mathscr{I}$-summable, we obtain then $\mathscr{J}\chi_X = \mathscr{I}(\chi_X f_0) = \mathscr{I}f_0$, and this shows that $\mathscr{J}\chi_X < \infty$ on account of the $\mathscr{I}$-summability of f_0.

(c) Since we have now that $\mathscr{I}(\chi_E f_0) = \mathscr{J}\chi_E = \mathscr{I}(\chi_E f_0{}^*)$ holds for any $\mathscr{I}$-measurable subset E of X, it follows easily that f_0 and $f_0{}^*$ are $\mathscr{I}$-almost equal.

6. *Kakutani's theorem for the infinite product integral*

As in Theorem 5.2, we consider the class of all Stieltjes–Lebesgue integrals having the linear vector lattice L as initial domain of definition, and inducing finite measures in X. If $\mathscr{I}$ and $\mathscr{J}$ are two of these integrals, and $\mathscr{J}$ is $\mathscr{I}$-absolutely continuous, we shall denote this by $\mathscr{J} \prec \mathscr{I}$. In this case the $\mathscr{I}$-summable function $f_0(x) \geqslant 0$, satisfying $\mathscr{J}f = \mathscr{I}(ff_0)$ for all $\mathscr{J}$-summable f, is sometimes denoted by $(d\mathscr{J}/d\mathscr{I})(x)$, and called a Radon-Nikodym derivative, as if $f_0(x)$ were a differential quotient. It is easy to see that if $\mathscr{J} \prec \mathscr{I} \prec \mathscr{K}$, then $d\mathscr{J}/d\mathscr{K}$ and the product $(d\mathscr{J}/d\mathscr{I})(d\mathscr{I}/d\mathscr{K})$ are $\mathscr{K}$-almost equal functions.

The integrals $\mathscr{I}$ and $\mathscr{J}$ (in the class referred to) are said to be orthogonal (or singular with respect to each other) whenever there exists a decomposition of X into disjoint sets E and F such that $\mathscr{I}\chi_F = \mathscr{J}\chi_E = 0$. The notation for this is $\mathscr{I} \perp \mathscr{J}$.

Assume now that $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{K}$ are integrals in the class referred to such that $\mathscr{I} \prec \mathscr{K}$ and $\mathscr{J} \prec \mathscr{K}$. The induced measures will be denoted by μ_I, μ_J and μ_K respectively. The functions $(d\mathscr{I}/d\mathscr{K})^{1/2}$ and $(d\mathscr{J}/d\mathscr{K})^{1/2}$ are evidently quadratically $\mathscr{K}$-summable, so by Schwarz's inequality the integral

$$\varrho(\mathscr{I}, \mathscr{J}) = \int \{(d\mathscr{I}/d\mathscr{K})(d\mathscr{J}/d\mathscr{K})\}^{1/2}\, d\mu_K \tag{1}$$

is finite. Let $\mathscr{P}$ be another integral in the class referred to such that $\mathscr{I} \prec \mathscr{P}$ and $\mathscr{J} \prec \mathscr{P}$. Then

$$\int \{(d\mathscr{I}/d\mathscr{P})(d\mathscr{J}/d\mathscr{P})\}^{1/2}\, d\mu_P = \int \{(d\mathscr{I}/d\mathscr{K})(d\mathscr{J}/d\mathscr{K})\}^{1/2}\, d\mu_K. \tag{2}$$

The number $\varrho(\mathscr{I}, \mathscr{J})$ is independent, therefore, of the choice of the majorant $\mathscr{K}$. For the proof, let $\mathscr{Q} = \mathscr{K} + \mathscr{P}$. Then $\mathscr{K} \prec \mathscr{Q}$ and $\mathscr{P} \prec \mathscr{Q}$, and the right side of (2) equals

$$\begin{gathered}\int \{(d\mathscr{I}/d\mathscr{K})(d\mathscr{J}/d\mathscr{K})\}^{1/2}\, (d\mathscr{K}/d\mathscr{Q}) d\mu_Q = \\ \int \{(d\mathscr{I}/d\mathscr{K})(d\mathscr{K}/d\mathscr{Q})\}^{1/2} \{(d\mathscr{J}/d\mathscr{K})(d\mathscr{K}/d\mathscr{Q})\}^{1/2}\, d\mu_Q = \\ \int \{(d\mathscr{I}/d\mathscr{Q})(d\mathscr{J}/d\mathscr{Q})\}^{1/2}\, d\mu_Q.\end{gathered}$$

Similarly for the left side of (2), and the equality in (2) follows.

Given any pair $\mathscr{I}$, $\mathscr{J}$ in the class referred to, there always exists an integral $\mathscr{K}$ in the same class such that $\mathscr{I} \prec \mathscr{K}$ and $\mathscr{J} \prec \mathscr{K}$; hence, $\varrho(\mathscr{I}, \mathscr{J})$ is always defined. The simplest choice for $\mathscr{K}$ is $\mathscr{K} = \mathscr{I} + \mathscr{J}$. Furthermore, it follows by Schwarz's inequality that

$$\varrho(\mathscr{I}, \mathscr{J}) \leqslant \{\mathscr{I}\chi_X \cdot \mathscr{J}\chi_X\}^{1/2}.$$

Finally, $\mathscr{I}$ and $\mathscr{J}$ are orthogonal if and only if $\varrho(\mathscr{I}, \mathscr{J}) = 0$. Indeed, if $\mathscr{I} \perp \mathscr{J}$, there exists a decomposition of X into disjoint sets E and F such that $\mathscr{I}\chi_F = \mathscr{J}\chi_E = 0$. Then, if $\mathscr{K}$ is a common majorant, we have $d\mathscr{I}/d\mathscr{K} = 0$ $\mathscr{K}$-almost everywhere on F, and $d\mathscr{J}/d\mathscr{K} = 0$ $\mathscr{K}$-almost everywhere on E. Hence, $\varrho(\mathscr{I}, \mathscr{J}) = 0$. Conversely, if $\varrho(\mathscr{I}, \mathscr{J}) = 0$, it follows that $(d\mathscr{I}/d\mathscr{K})(d\mathscr{J}/d\mathscr{K})$ is $\mathscr{K}$-almost every zero. Hence, if $E = \{x : d\mathscr{I}/d\mathscr{K} \neq 0\}$ and $F = X - E$, then $d\mathscr{I}/d\mathscr{K} = 0$ on F and $d\mathscr{J}/d\mathscr{K} = 0$ $\mathscr{K}$-almost everywhere on E, so $\mathscr{I}\chi_F = \mathscr{J}\chi_E = 0$.

Assume now that $X_i (i = 1, \ldots, n)$ are non-empty point sets and, for each i, let $\mathscr{I}_i f$ be a Stieltjes-Lebesgue integral over X_i, having the linear vector lattice $L_{(i)}$ of real bounded functions on X_i as initial domain of definition, and inducing the finite measure μ_i in X_i. According to the usual method, the product integral $\mathscr{I} f = \mathscr{I}_1 \ldots \mathscr{I}_n f$ over the Cartesian product $X_1 \times \ldots \times X_n$ is first defined on the linear vector lattice of all step functions of the form $f(x) = \sum_{k=1}^{p} c_k \chi_{C_k}(x)$ with $C_k = A_{1k} \times \ldots \times A_{nk}$ such that A_{ik} is μ_i-measurable, and for this $f(x)$ the definition is then that

$\mathscr{I}f = \sum_{k=1}^{p} c_k \prod_{i=1}^{n} \mu_i(A_{ik})$. By applying the extension procedure to this elementary integral the extended integral $\mathscr{I}f$ is obtained. It follows easily from the results in the preceding section that the same product integral is obtained if the sets $A_{ik} \subset X_i$ are restricted to the Borel sets (with respect to $L_{(i)}$) in X_i. Indeed, by Theorem 5.1 any μ_i-measurable set in X_i is μ_i-almost equal to a Borel set. Hence, if $\mathscr{I}f$ is taken in the restricted sense, then any set $C = A_1 \times \ldots \times A_n$, where $A_i \subset X_i$ is μ_i-measurable but not necessarily a Borel set, becomes $\mathscr{I}$-measurable with $\mathscr{I}\chi_C = \prod_1^n \mu_i(A_i)$.

Next, assume again that the points sets $X_i (i = 1, \ldots, n)$ are non-empty and, for each i, let $\mathscr{I}_i f$ and $\mathscr{J}_i f$ be Stieltjes-Lebesgue integrals over X_i, having the linear vector lattice $L_{(i)}$ as initial domain of definition, and inducing finite positive measures in X_i. The product integrals $\mathscr{I} = \mathscr{I}_1 \ldots \mathscr{I}_n$ and $\mathscr{J} = \mathscr{J}_1 \ldots \mathscr{J}_n$ may then be regarded as extensions of elementary integrals on the domain of definition L, where L consists of all step functions $f(x) = \sum_1^p c_k \chi_{C_k}(x)$ with $C_k = A_{1k} \times \ldots \times A_{nk}$ such that A_{ik} is a Borel set with respect to $L_{(i)}$. Let $\mathscr{K}_i (i = 1, \ldots, n)$ be an auxiliary integral over X_i such that $\mathscr{I}_i \prec \mathscr{K}_i$, $\mathscr{J}_i \prec \mathscr{K}_i$, and set $\mathscr{K} = \mathscr{K}_1 \ldots \mathscr{K}_n$. It follows then immediately that the function $p = \prod_1^n d\mathscr{I}_i / d\mathscr{K}_i$ satisfies $\mathscr{I}f = \mathscr{K}(fp)$ for all $f \in L$, and by Theorem 5.3 the equality $\mathscr{I}f = \mathscr{K}(fp)$ is preserved after extension of $\mathscr{I}$ and $\mathscr{K}$. Similarly, $\mathscr{J}f = \mathscr{K}(fq)$ holds for $q = \prod_1^n d\mathscr{J}_i / d\mathscr{K}_i$ and any $\mathscr{J}$-summable f. Hence $\varrho(\mathscr{I}, \mathscr{J}) = \prod_1^n \varrho(\mathscr{I}_i, \mathscr{J}_i)$, where ϱ is defined as in (1). Evidently $\varrho(\mathscr{I}, \mathscr{J}) = 0$ if and only if $\varrho(\mathscr{I}_i, \mathscr{J}_i) = 0$ for at least one value of i, that is, $\mathscr{I} \perp \mathscr{J}$ if and only if $\mathscr{I}_i \perp \mathscr{J}_i$ for at least one value of i.

Furthermore, in the case that $\mathscr{J}_i \prec \mathscr{I}_i$ for all i, we may select $\mathscr{K}_i = \mathscr{I}_i$ for all i, so $\mathscr{K} = \mathscr{I}$. The equality $\mathscr{J}f = \mathscr{K}(fq)$ with $q = \prod_1^n d\mathscr{J}_i / d\mathscr{K}_i$ becomes then $\mathscr{J}f = \mathscr{I}(fq)$ with $d\mathscr{J}/d\mathscr{I} = q = \prod_1^n d\mathscr{J}_i / d\mathscr{I}_i$. Conversely, if $\mathscr{J} \prec \mathscr{I}$, then $\mathscr{J}_i \prec \mathscr{I}_i$ for all i. In order to show, e.g., that $\mathscr{J}_1 \prec \mathscr{I}_1$, let $A_1 \subset X_1$ be an $\mathscr{I}_1$-null set. Then $P = A_1 \times X_2 \times \ldots \times X_n$ satisfies $\mathscr{I}\chi_P = 0$, so $\mathscr{J}\chi_P = 0$ by hypothesis. This implies that χ_P is $\mathscr{J}$-summable, so $0 = \mathscr{J}\chi_P = (\mathscr{J}_1 \chi_{A_1})(\mathscr{J}_2 \chi_{X_2}) \ldots (\mathscr{J}_n \chi_{X_n})$ by Fubini's theorem, and since $\mathscr{J}_i \chi_{X_i} > 0$ by hypothesis, it follows that $\mathscr{J}_1 \chi_{A_1} = 0$.

In order to state Kakutani's theorem on infinite product integrals we assume that $X_i (i = 1, 2, \ldots)$ is an infinite sequence of non-empty point sets and, for each i, we let $\mathscr{I}_i f$ and $\mathscr{J}_i f$ be Stieltjes–Lebesgue integrals over X_i, having the linear vector lattice $L_{(i)}$ as initial domain of definition, and satisfying $\mathscr{I}_i \chi_{X_i} = \mathscr{J}_i \chi_{X_i} = 1$. It follows immediately that

$$\varrho(\mathscr{I}_i, \mathscr{I}_i) = \varrho(\mathscr{J}_i, \mathscr{J}_i) = 1 \quad \text{and} \quad 0 \leqslant \varrho(\mathscr{I}_i, \mathscr{J}_i) \leqslant 1$$

for all i. According to the usual method, the product integral $\mathscr{I}f$ of the integrals $\mathscr{I}_i f$ over the infinite Cartesian product $X_\omega = X_1 \times X_2 \times \ldots$ is first defined on the linear vector lattice of all step functions $f(x) = \sum_{k=1}^{p} c_k \chi_{C_k}(x)$ with $C_k = A_{1k} \times \ldots \times A_{n_k, k} \times X_{n_k+1} \times \ldots$ such that $A_{ik} \subset X_i$

is a Borel set (with respect to $L_{(i)}$), and for this f the definition is then that $\mathscr{I}f = \sum_{k=1}^{p} c_k \prod_{i=1}^{n_k} \mu_i(A_{ik})$. By applying the extension procedure to this elementary integral the extended $\mathscr{I}f$ is obtained. The product integral $\mathscr{J}f$ of the integrals $\mathscr{J}_i f$ is defined similarly.

Theorem 6.1 (S. Kakutani, [2]). *Let $\mathscr{I}f$ and $\mathscr{J}f$ be the product integrals of the integrals $\mathscr{I}_i f$ and $\mathscr{J}_i f$ respectively, and let $\mathscr{J}_i \prec \mathscr{I}_i$ for all $i = 1, 2, \ldots$. Then $\mathscr{J} \prec \mathscr{I}$ if $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i) > 0$, and $\mathscr{J} \perp \mathscr{I}$ if $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i) = 0$. In either case $\varrho(\mathscr{I}, \mathscr{J}) = \prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i)$.*

Proof. The main points of the proof will be recalled. We have $\mathscr{Q}_n = \mathscr{J}_1 \ldots \mathscr{J}_n \prec \mathscr{I}_1 \ldots \mathscr{I}_n = \mathscr{P}_n$ by what has been observed above, and the functions $\psi_n(x) = (d\mathscr{Q}_n/d\mathscr{P}_n)^{1/2} = (\prod_{i=1}^n d\mathscr{J}_i/d\mathscr{I}_i)^{1/2}$, regarded as functions of $x = (x_1, x_2, \ldots) \in X_\omega = X_1 \times X_2 \times \ldots$, are elements of the Hilbert space L_2 (integration with respect to the integral $\mathscr{I}$) such that $\|\psi_n\|^2 = \mathscr{I}(\psi_n^2) = 1$ and $\|\psi_m - \psi_n\|^2 = 2 - 2\prod_{n+1}^m \varrho(\mathscr{I}_i, \mathscr{J}_i)$ for $m > n$. Hence, if $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i) > 0$, the functions ψ_n form a fundamental sequence in the Hilbert space referred to, and the limit function ψ is easily shown to satisfy $\mathscr{J}f = \mathscr{I}(f\psi^2)$. This implies that $\mathscr{J} \prec \mathscr{I}$, and $d\mathscr{J}/d\mathscr{I} = \psi^2$. Hence

$$\varrho(\mathscr{I}, \mathscr{J}) = \mathscr{I}\psi = \lim \mathscr{I}\psi_n = \lim \mathscr{I}_1 \ldots \mathscr{I}_n \psi_n = \lim \prod_1^n \varrho(\mathscr{I}_i, \mathscr{J}_i).$$

If $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i) = 0$, and $\varepsilon > 0$ is given, there is an index k such that $\prod_1^k \varrho(\mathscr{I}_i, \mathscr{J}_i) < \varepsilon$. Let $B = \{x: \prod_1^k d\mathscr{J}_i/d\mathscr{I}_i > 1\}$. Then it is easily shown that $\mathscr{I}\chi_B < \varepsilon$ and $\mathscr{J}\chi_{X_\omega - B} < \varepsilon$. If now $\varepsilon_n = 2^{-n}$ and B_n is such that $\mathscr{I}\chi_{B_n} < \varepsilon$ and $\mathscr{J}\chi_{X_\omega - B_n} < \varepsilon$, then $B = \lim\sup B_n$ satisfies $\mathscr{I}\chi_B = 0$ and $\mathscr{J}\chi_{X_\omega - B} = 0$. This shows that $\mathscr{I} \perp \mathscr{J}$, so $\varrho(\mathscr{I}, \mathscr{J}) = 0 = \prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i)$.

There exists a generalization of part of Kakutani's theorem for the case that $\mathscr{J}_i \prec \mathscr{I}_i$ does not necessarily hold for all i. In this case, as well as in the case that we have $\mathscr{J}_i \prec \mathscr{I}_i$ for all i, $\varrho(\mathscr{I}, \mathscr{J})$ and the infinite product $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i)$ exist as finite numbers, and one may ask, therefore, whether the equality $\varrho(\mathscr{I}, \mathscr{J}) = \prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i)$ continues to hold. We shall prove that the answer is affirmative, and for this purpose we first present a simple lemma.

Lemma 6.2. *Let $\mathscr{I}$ and $\mathscr{J}$ be Stieltjes–Lebesgue integrals over X, having the same linear vector lattice as initical domain of definition, and satisfying $\mathscr{I}\chi_X = \mathscr{J}\chi_X = 1$. Let $0 < \alpha < 1$, and set $\mathscr{K} = (1-\alpha)\mathscr{I} + \alpha\mathscr{J}$. Then $\mathscr{K}\chi_X = 1$, $\mathscr{I} \prec \mathscr{K}$ and $\mathscr{J} \prec \mathscr{K}$, and $(d\mathscr{I}/d\mathscr{K})(x) \leqslant (1-\alpha)^{-1}$ for $\mathscr{K}$-almost every x. Furthermore $\varrho(\mathscr{I}, \mathscr{J}) \leqslant (1-\alpha)^{-1/2} \varrho(\mathscr{J}, \mathscr{K})$ and $\varrho(\mathscr{I}, \mathscr{K}) \geqslant 1 - \alpha$.*

Proof. Evidently $\mathscr{K}\chi_X = 1$, and $\mathscr{I} \prec \mathscr{K}$ as well as $\mathscr{J} \prec \mathscr{K}$. Since $(1-\alpha)(d\mathscr{I}/d\mathscr{K})(x) + \alpha(d\mathscr{J}/d\mathscr{K})(x) = 1$ for $\mathscr{K}$-almost every x, we have $(d\mathscr{I}/d\mathscr{K})(x) \leqslant (1-\alpha)^{-1}$ for $\mathscr{K}$-almost every x. It follows immediately that

$$\begin{aligned}\varrho(\mathscr{I}, \mathscr{J}) &= \mathscr{K}\{(d\mathscr{I}/d\mathscr{K})^{1/2}(d\mathscr{J}/d\mathscr{K})^{1/2}\} \leqslant \\ &\leqslant (1-\alpha)^{-1/2}\mathscr{K}\{(d\mathscr{J}/d\mathscr{K})^{1/2}\} = (1-\alpha)^{-1/2}\varrho(\mathscr{J}, \mathscr{K}).\end{aligned}$$

For the proof of $\varrho(\mathscr{I}, \mathscr{K}) \geqslant 1-\alpha$, let $A = \{x : d\mathscr{I}/d\mathscr{K} > 1\}$ and $B = X - A = \{x : d\mathscr{I}/d\mathscr{K} \leqslant 1\}$. Then

$$\begin{aligned}\varrho(\mathscr{I}, \mathscr{K}) &= \mathscr{K}\{(d\mathscr{I}/d\mathscr{K})^{1/2}\} \geqslant \mathscr{K}\chi_A + \mathscr{K}\{\chi_B(d\mathscr{I}/d\mathscr{K})\} = \\ &= \mathscr{K}\chi_A + \mathscr{I}\chi_B = (1-\alpha)\,\mathscr{I}\chi_A + \alpha\mathscr{J}\chi_A + (1-\alpha)\,\mathscr{I}\chi_B + \alpha\mathscr{I}\chi_B = \\ &= (1-\alpha)\,\mathscr{I}\chi_X + \alpha(\mathscr{J}\chi_A + \mathscr{I}\chi_B) = \\ &= 1-\alpha + \alpha(\mathscr{J}\chi_A + \mathscr{I}\chi_B) \geqslant 1-\alpha.\end{aligned}$$

Theorem 6.3. *Let $\mathscr{I}$ and $\mathscr{J}$ be the product integrals of the integrals $\mathscr{I}_i f$ and $\mathscr{J}_i f$ respectively, where $\mathscr{I}_i \chi_{X_i} = \mathscr{J}_i \chi_{X_i} = 1$ for all i. Then $\varrho(\mathscr{I}, \mathscr{J}) = \prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i)$.*

Proof. Select a sequence of numbers $\alpha_i (i = 1, 2, \ldots)$ such that $0 < \alpha_i < 1$ and $\sum_1^\infty \alpha_i < \infty$, and set $\mathscr{K}_i = (1-\alpha_i)\mathscr{I}_i + \alpha_i \mathscr{J}_i$. Since $1 - \alpha_i \leqslant \varrho(\mathscr{I}_i, \mathscr{K}_i) \leqslant 1$ by the preceding lemma, and since $\prod_1^\infty (1-\alpha_i) > 0$ on account of $\sum_1^\infty \alpha_i < \infty$, we have $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{K}_i) > 0$. It follows by Kakutani's theorem that $\mathscr{I} \prec \mathscr{K}$, where $\mathscr{K}$ is the product integral of the integrals $\mathscr{K}_i$.

Since $\mathscr{J}_i \prec \mathscr{K}_i$ for all i, there are two possibilities by Kakutani's theorem: either $\mathscr{J} \prec \mathscr{K}$ or $\mathscr{J} \perp \mathscr{K}$.

In the case that $\mathscr{J} \prec \mathscr{K}$, we set

$$\varphi_n = \{d(\mathscr{I}_1 \ldots \mathscr{I}_n)/d(\mathscr{K}_1 \ldots \mathscr{K}_n)\}^{1/2},$$
$$\psi_n = \{d(\mathscr{J}_1 \ldots \mathscr{J}_n)/d(\mathscr{K}_1 \ldots \mathscr{K}_n)\}^{1/2}.$$

Then φ_n and ψ_n converge in the Hilbert space L_2 (integration with respect to $\mathscr{K}$) to $\varphi = (d\mathscr{I}/d\mathscr{K})^{1/2}$ and $\psi = (d\mathscr{J}/d\mathscr{K})^{1/2}$ respectively, so

$$\varrho(\mathscr{I}, \mathscr{J}) = (\varphi, \psi) = \lim\,(\varphi_n, \psi_n) = \lim \prod_1^n \varrho(\mathscr{I}_i, \mathscr{J}_i).$$

In the case that $\mathscr{J} \perp \mathscr{K}$, it follows from $\mathscr{I} \prec \mathscr{K}$ that $\mathscr{I} \perp \mathscr{J}$, so $\varrho(\mathscr{I}, \mathscr{J}) = 0$, and it remains to prove that $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i) = 0$. Observe first that $(1-\alpha_i)^{-1/2} \leqslant (1+2\alpha_i)^{1/2}$, so $\prod_1^\infty (1-\alpha_i)^{-1/2}$ is finite in view of the convergence of $\sum_1^\infty \alpha_i$. Hence, since $\prod_1^\infty \varrho(\mathscr{J}_i, \mathscr{K}_i) = 0$ on account of $\mathscr{J} \perp \mathscr{K}$, and $\varrho(\mathscr{I}_i, \mathscr{J}_i) \leqslant (1-\alpha_i)^{-1/2} \varrho(\mathscr{J}_i, \mathscr{K}_i)$ by the preceding lemma, it follows that $\prod_1^\infty \varrho(\mathscr{I}_i, \mathscr{J}_i) = 0$.

(To be continued).